NORTH-HOLLAND

Multirational Function Approximation via Linear Programming

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ABSTRACT

This paper presents a new method for the approximation of multi-real rational functions via linear programming. The formulation of the linear approximation problem is based on the minimization of a suitable criterion, which arrives from the well-known minimax criterion. The proposed algorithm permits the simultaneous approximation of the rational functions and has significant applications in many engineering design problems. Analytical examples are presented to illustrate the effectiveness and the advantages of the algorithm.

1. INTRODUCTION

Many engineering design problems require the simultaneous approximation of real rational functions. For example, in the digital filter design area [1–2], it is possible to design IIR digital filters by using a parallel configuration, similar to that of Figure 1. In the one-dimensional case, we want to approximate an ideal function (usually data points of the response of a system) by a real rational function. Many algorithms have been proposed for the one-dimensional case, such as Pade approximation [3] and the Differential Correction Algorithm (DCA) [4]. A fast linear programming method for

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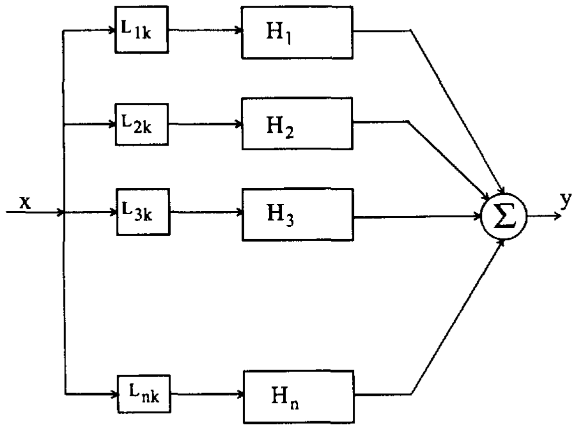

FIG. 1. Parallel system configuration.

the approximation of real rational functions (RFA algorithm) has been proposed by Papamarkos et al. [5–6]. Applications of this algorithm for the design of IIR digital filters are described in [7–8].

This paper proposes a new method for simultaneous approximation of multi-real rational functions (MRFAs) via linear programming. The new method can handle simultaneously multi-real rational functions, and the final approximation problem is formulated as a linear programming problem. It should be pointed out that the use of linear programming gives serious advantages and, mainly, the guaranteed convergence of the algorithm to the global solution. Additionally, it is ensured that the approximation problem always has solution, and this solution is derived by using the Revised Simplex Algorithm (RSA) [9].

2. DESCRIPTION OF THE ALGORITHM

The problem is formulated as follows. Let $H_i(x)$, $i = 1, 2, \ldots, n$ real rational functions of the following general form:

$$H_i(x) = \frac{A_i(x)}{B_i(x)} = \frac{\sum_{j=1}^{N_i} a_{ij} \tilde{A}_{ij}(x)}{1 + \sum_{j=1}^{M_i} b_{ij} \tilde{B}_{ij}(x)} \tag{1}$$

where N_j and M_j, $i = 1, 2, \ldots, n$ are given integers, x is the vector of the independent variables, a_{ij} and b_{ij} the unknown coefficients to be determined, and $\tilde{A}_{ij}(x)$, $\tilde{B}_{ij}(x)$ known functions of x.

Let also the independent variables x belong to a sampling region S, i.e., $x_k \in S$ with $k = 1, 2, \ldots, K$, where K is the total number of sampling points. The aim of the proposed algorithm is to determine the n unknown rational functions $H_i(x)$ such as to satisfy at every sampling point k the following criterion:

$$|G(x_k) - L_{1k} H_1(x_k) - L_{2k} H_2(x_k) \cdots - L_{nk} H_n(x_k)| \leqslant \delta, \qquad (2)$$

where L_{ik} are known real coefficients and δ is a sufficiently small positive value.

To solve the above approximation problem we will follow a separate procedure. Specifically, for each rational function $H_i(x)$, $i = 1, \ldots, n$, and for every sampling point $x_k \in S$, the following differences are considered:

$$|c_i G(x_k) B_i(x_k) - A_i(x_k)| = \xi'_{ik}, \qquad (3)$$

where c_i, $i = 1, 2, \ldots, n$ are unknown variables such that

$$\left.\begin{array}{r} 0 \leqslant c_i \leqslant 1 \\ \sum_{i=1}^{n} c_i = 1 \end{array}\right\}. \qquad (4)$$

Since we accept that $B_i(x_k) > 0$ for every $x_k \in S$, the relation (3) may also be written in the form:

$$\left| c_i G(x_k) - \frac{A_i(x_k)}{B_i(x_k)} \right| \leqslant \frac{\xi'_{ik}}{B_i(x_k)}. \qquad (5)$$

From equation (5), it is clear that for a satisfactory approximation the quantities

$$\delta'_{ik} = \frac{B_i(x_k)}{\xi'_{ik}} \qquad (6)$$

must achieve large positive values. If, therefore,

$$\xi' = \underset{x_k \in S}{\text{maximum}}\{\xi'_{ik}\} \qquad (7)$$

and

$$\frac{c_i}{\delta_i} = \max_{x_k \in S} \text{imum} \left\{ \frac{\xi'}{B_{ik}} \right\}, \tag{8}$$

then, the approximation problem may be formulated as follows: maximize $\delta_1 + \delta_2 + \cdots + \delta_n$ subject to the constraints

$$\begin{gathered} | c_i G(x_k) B_i(x_k) - L_{ik} A_i(x_k) | \leqslant \xi', \\ \frac{\xi'}{B_i(x_k)} \leqslant \frac{c_i}{\delta_i}, \qquad c_1 + c_2 + \cdots + c_n = 1 \\ \text{for } k = 1, 2, \ldots, K \quad \text{and} \quad i = 1, 2, \ldots, n, \end{gathered} \tag{9}$$

where $c_i \geqslant 0$, $\xi' \geqslant 0$ and $\delta_i > 0$.

From the above formulation of the approximation problem we can conclude that:

- For every $x_k \in S$ and for every i the following relations always hold:

$$\left| c_i G(x_k) - L_{ik} \frac{A_i(x_k)}{B_i(x_k)} \right| \leqslant \frac{\xi'}{B_i(x_k)} \leqslant \frac{c_i}{\delta_i} \leqslant \frac{1}{\delta_i}. \tag{10}$$

Consequently, the final optimal solution guarantees that

$$\left| G(x_k) - \sum_{i=1}^{n} L_{ik} H_i(x_k) \right| \leqslant \sum_{i=1}^{n} | c_i G(x_k) - L_{ik} H_i(x_k) | \leqslant \sum_{i=1}^{n} \frac{c_i}{\delta_i}. \tag{11}$$

- The second set of constraints guarantees the positiveness of $B_i(x_k)$ for every $x_k \in S$.
- The approximation problem (9) does not have a linear form.

By using equation (1), the approximation problem (9) is rewritten in the following form: maximize $\delta_1 + \delta_2 + \cdots + \delta_n$ subject to

$$\begin{gathered} \left| c_i G(x_k) \left(1 + \sum_{j=1}^{M_i} b_{ij} \tilde{B}_{ij}(x_k) \right) - L_{ik} \left(\sum_{j=1}^{N_i} a_{ij} \tilde{A}_{ij}(x_k) \right) \right| \leqslant \xi', \\ -c_i \left(1 + \sum_{j=1}^{M_i} b_{ij} \tilde{B}_{ij}(x_k) \right) + \xi' \delta_i \leqslant 0, \qquad c_1 + c_2 + \cdots + c_n = 1 \\ \text{for } k = 1, 2, \ldots, K \quad \text{and} \quad i = 1, 2, \ldots, n \end{gathered} \tag{12}$$

where $c_i \geqslant 0$, $\xi' \geqslant 0$ and $\delta_i > 0$.

The approximation problem, as formulated in equation (12), is not linear, but it may be converted to a linear one via the transformations

$$\left.\begin{aligned} b'_{i0} &= \frac{c_i}{\xi'}, && \text{for } i = 1,2,\dots,n \\ b'_{ij} &= \frac{c_i b_{ij}}{\xi'}, && \text{for } i = 1,2,\dots,n \text{ and } j = 1,2,\dots,M_i \\ d'_{ij} &= \frac{a_{ij}}{\xi'}, && \text{for } i = 1,2,\dots,n \text{ and } j = 1,2,\dots,N_i \\ \xi &= \frac{1}{\xi'} \end{aligned}\right\}. \tag{13}$$

Now, using (13), the approximation problem is reformulated as: maximize $\delta_1 + \delta_2 + \cdots + \delta_n$ subject to

$$\begin{aligned} G_k\left[b'_{i0} + \sum_{j=1}^{M_i} b'_{ij}\tilde{B}_{ij}(x_k)\right] - L_{ik}\sum_{j=1}^{N_i} d'_{ij}\tilde{A}_{ij}(x_k) &\leqslant 1 \\ -G_k\left[b'_{i0} + \sum_{j=1}^{M_i} b'_{ij}\tilde{B}_{ij}(x_k)\right] + L_{ik}\sum_{j=1}^{N_i} d'_{ij}\tilde{A}_{ij}(x_k) &\leqslant 1 \\ -\left[b'_{i0} + \sum_{j=1}^{M_i} b'_{ij}\tilde{B}_{ij}(x_k)\right] + \delta_i &\leqslant 0 \end{aligned} \tag{14}$$

$$b'_{10} + b'_{20} + \cdots + b'_{n0} = \xi \qquad \text{for } k = 1,2,\dots,K \quad \text{and} \quad i = 1,2,\dots,n$$

where $b'_{i0} \geqslant 0$, $\xi \geqslant 0$ and $\delta_i > 0$.

It is noted that in the linear problem (14), the variables d'_{ij} and b'_{ij} are unrestricted in sing. On the other hand, linear programming algorithms, such as the RSA, require only nonnegative variables. A simple way to overcome this difficulty is to shift the variables by a suitable shift constant $V > 0$. Therefore, if we define that

$$\left.\begin{aligned} p_{ij} &= V + d'_{ij} \quad \text{for} \quad i = 1,2,\dots,n \text{ and } j = 1,2,\dots,N_i \\ q_{ij} &= V + b'_{ij} \quad \text{for} \quad i = 1,2,\dots,n \text{ and } j = 1,2,\dots,M_i \end{aligned}\right\}, \tag{15}$$

then the linear problem (14) is reformulated as maximize $\delta_1 + \delta_2 + \cdots + \delta_n$ subject to

$$G_k\left[b'_{i0} + \sum_{j=1}^{M_i} b'_{ij}\tilde{B}_{ij}(x_k)\right] - L_{ik}\sum_{j=1}^{N_i} d_{ij}\tilde{A}_{ij}(x_k) + d_{ik}\,y \leqslant 1$$

$$-G_k\left[b'_{i0} + \sum_{j=1}^{M_i} b'_{ij}\tilde{B}_{ij}(x_k)\right] + L_{ik}\sum_{j=1}^{N_i} d'_{ij}\tilde{A}_{ij}(x_k) - d_{ik}\,y \leqslant 1$$

$$-\left[b'_{i0} + \sum_{j=1}^{M_i} b'_{ij}\tilde{B}_{ij}(x_k)\right] + \delta_i + h_{ik}\,y \leqslant 0 \qquad (16)$$

$$b'_{10} + b'_{20} + \cdots + b'_{n0} = \xi$$

$$y = 1 \quad \text{for } k = 1, 2, \ldots, K \quad \text{and} \quad i = 1, 2, \ldots, n$$

where all the variables are positives, and

$$d_{ik} = -G_k V \sum_{j=1}^{N_i} \tilde{B}_{ij}(x_k) + L_{ik} V \sum_{j=1}^{M_i} \tilde{A}_{ij}(x_k) \qquad (17)$$

$$h_{ik} = V \sum_{j=1}^{N_i} \tilde{B}_{ij}(x_k). \qquad (18)$$

The final formulation of the approximation problem corresponds to a total number of $\sum_{i=1}^{n}[N(i) + M(i)] + n + 2$ variables and $3Kn + 2$ linear constraints. Therefore, for a large value of K, the approximation problem must be solved by its dual, using a method such as the RSA.

3. EXAMPLES

EXAMPLE 1. In this first example, we use the MRFA algorithm for the approximation of the $G(x) = \sqrt{x}$ function by two rational functions of the following forms:

$$H_1(x) = \frac{\sum_{i=1}^{3} a_{i1} x^{i-1}}{1 + \sum_{i=1}^{3} b_{i1} x^{i}}, \qquad H_2(x) = \frac{\sum_{i=1}^{3} a_{i2} x^{i-1}}{1 + \sum_{i=1}^{3} b_{i2} x^{i}}. \qquad (19)$$

The sampling interval is $[0, 1]$, 28 equal spaced sampling points were used, and the shift value is equal to $V = 10$. Table 1 gives the values of the coefficients L_{1k} and L_{2k} for every sampling point k. The problem is solved from its dual by using the RSA, and the global solution was found after 12.35 sec. It should be noted that the same problem was solved by using the linear programming procedure of the Mathematica 2.1, and the results reached were exactly the same.

Table 2 summarizes the complete results for this example, while Figures 2 and 3 show the fitting results and the approximation errors, respectively. According to the values of Table 2 and the relations (10) and (11), we can observe that

$$|c_1 G(x_k) - L_{1k} H_1(x_k)| \leqslant \frac{c_1}{\delta_1} = 0.09915 \tag{20}$$

$$|c_2 G(x_k) - L_{2k} H_2(x_k)| \leqslant \frac{c_2}{\delta_2} = 0.0873 \tag{21}$$

and

$$|G(x_k) - L_{1k} H_1(x_k) - L_{2k} H_2(x_k)| \leqslant \frac{c_1}{\delta_1} + \frac{c_2}{\delta_2} = 0.1864. \tag{22}$$

EXAMPLE 2. This example describes the approximation of the MRFA algorithm for simultaneous approximation of the function $G(x) = 4 +$

TABLE 1

VALUES OF L_{1k} AND L_{2k}

k	L_{1k}	L_{2k}	k	L_{1k}	L_{2k}	k	L_{1k}	L_{2k}	k	L_{1k}	L_{2k}
1	0.4	0.6	8	0.5	0.5	15	0.5	0.5	22	0.6	0.4
2	0.3	0.7	9	0.5	0.5	16	0.45	0.55	23	0.62	0.38
3	0.5	0.5	10	0.6	0.4	17	0.45	0.55	24	0.55	0.45
4	0.6	0.4	11	0.65	0.35	18	0.5	0.5	25	0.5	0.5
5	0.36	0.64	12	0.5	0.5	19	0.55	0.45	26	0.5	0.5
6	0.7	0.3	13	0.65	0.35	20	0.5	0.5	27	0.5	0.5
7	0.4	0.6	14	0.6	0.4	21	0.4	0.6	28	0.4	0.4

TABLE 2
RESULTS FOR EXAMPLE 1

First rational function		Second rational function	
$\xi = 11.464486$			
$\delta_1 = 5.689248$	$\frac{c_1}{\delta_1} = 0.09915$	$\delta_2 = 4.992666$	$\frac{c_2}{\delta_2} = 0.0873$
$b'_{10} = 6.467268$	$c_1 = 0.564113$	$b'_{20} = 4.997218$	$c_2 = 0.435887$
$p_{11} = 11.900672$	$a_{11} = 0.165788$	$p_{21} = 11.666667$	$a_{21} = 0.145377$
$p_{12} = 27.136498$	$a_{12} = 1.494746$	$p_{22} = 23.541586$	$a_{22} = 1.181177$
$p_{13} = 1.155346$	$a_{13} = -0.771483$	$p_{23} = 4.773414$	$a_{23} = -0.455894$
$q_{11} = 11.991472$	$b_{11} = 0.307931$	$q_{21} = 9.972362$	$b_{21} = -0.005531$
$q_{12} = 0.0000000$	$b_{12} = -1.546248$	$q_{22} = 10.052326$	$b_{22} = 0.010471$
$q_{13} = 17.956319$	$b_{13} = 1.230244$	$q_{23} = 9.970760$	$b_{23} = -0.005851$

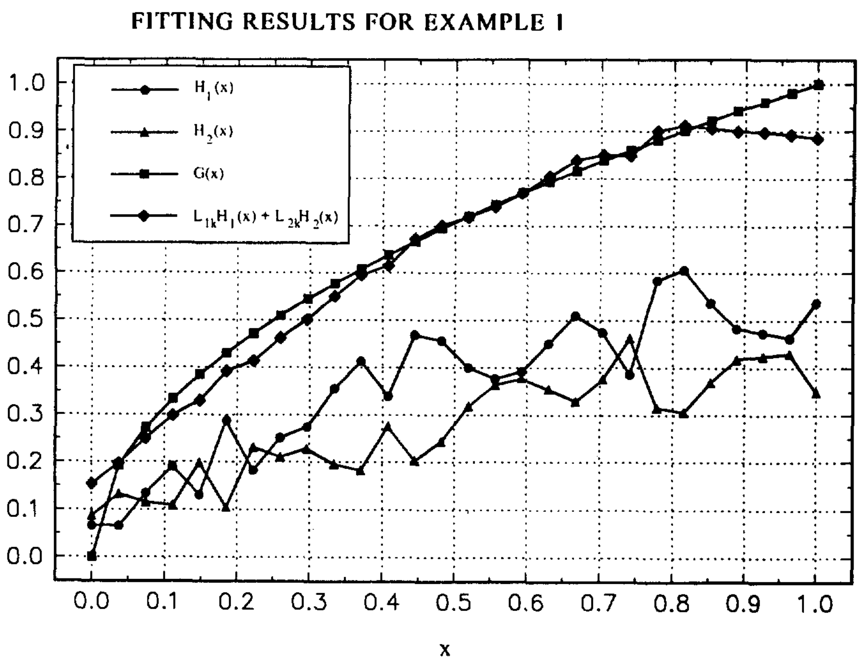

FIG. 2. Approximation of $G(x) = \sqrt{x}$ by two rational functions.

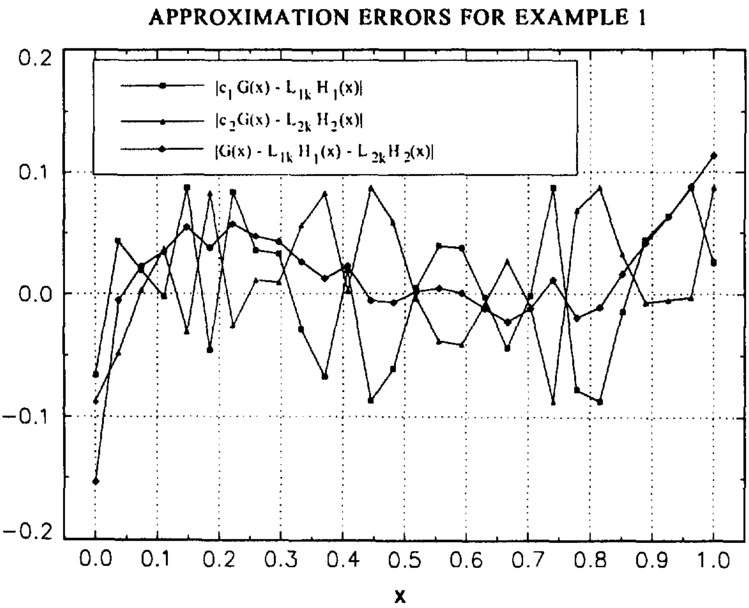

FIG. 3. Approximation errors for Example 1.

$10\,e^{-10x} + 2\,e^{-5x}$ via three rational functions of the following forms:

$$H_1(x) = \frac{\sum_{i=1}^{3} a_{i1}\,x^{i-1}}{1 + \sum_{i=1}^{3} b_{i1}\,x^{i}}, \qquad H_2(x) = \frac{\sum_{i=1}^{3} a_{i2}\,x^{i-1}}{1 + \sum_{i=1}^{2} b_{i2}\,x^{i}},$$

and

$$H_3(x) = \frac{\sum_{i=1}^{2} a_{i3}\,x^{i-1}}{1 + \sum_{i=1}^{3} b_{i3}\,x^{i}}. \tag{23}$$

The problem is solved for $V = 10$ with 30 equal spaced sampling points in the interval $[0, 1]$.

For the lack of comparison, we use two different sets for the coefficients L_{ik}.

TABLE 3
RESULTS FOR $L_{ik} = 1$

First rational function	Second rational function	Third rational function
$\xi = 16.079569$		
$\delta_1 = 16.079569$	$\delta_2 = 6.798999$	$\delta_3 = 1.802961$
$c_1 = 0.465040$	$c_2 = 0.422835$	$c_3 = 0.112127$
$\frac{c_1}{\delta_1} = 0.02892$	$\frac{c_2}{\delta_2} = 0.06219$	$\frac{c_3}{\delta_3} = 0.06219$
$a_{11} = -0.431308$	$a_{21} = 6.703127$	$a_{31} = 3.338764$
$a_{12} = 78.483924$	$a_{22} = 0.042666$	$a_{32} = -0.621907$
$a_{13} = -0.621907$	$a_{23} = 73.519326$	$b_{31} = 9.750425$
$b_{11} = 5.788657$	$b_{21} = 5.757416$	$b_{32} = 1.856230$
$b_{12} = 38.612167$	$b_{22} = 40.497817$	$b_{33} = 0.327567$
$b_{13} = 7.378456$		

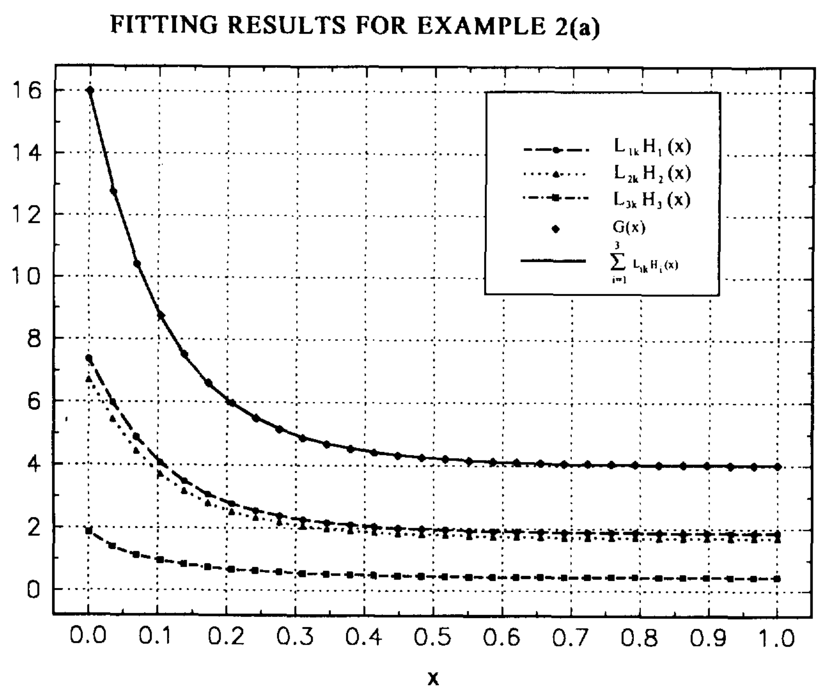

FIG. 4. Approximation of $G(x) = 4 + 10e^{-10x} + 2e^{-5x}$ by three rational functions and with $L_{ik} = 1$.

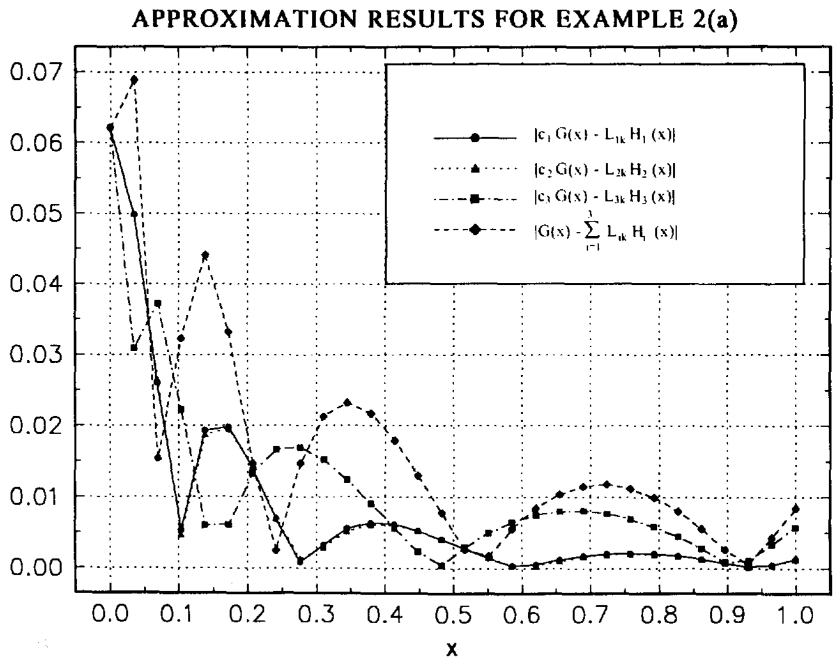

FIG. 5. Approximation errors for Example 2(a).

TABLE 4
VALUES OF L_{ik} FOR THE SECOND CASE OF EXAMPLE 2

k	L_{1k}	L_{2k}	L_{3k}	k	L_{1k}	L_{2k}	L_{3k}
1	0.27	0.2	0.27	16	0.25	0.22	0.23
2	0.3	0.23	0.26	17	0.29	0.26	0.27
3	0.23	0.22	0.21	18	0.21	0.25	0.24
4	0.29	0.29	0.27	19	0.25	0.28	0.25
5	0.22	0.23	0.21	20	0.24	0.27	0.25
6	0.21	0.26	0.28	21	0.26	0.24	0.2
7	0.25	0.24	0.24	22	0.27	0.21	0.26
8	0.23	0.25	0.25	23	0.29	0.25	0.27
9	0.22	0.23	0.27	24	0.25	0.24	0.28
10	0.23	0.28	0.3	25	0.26	0.24	0.3
11	0.29	0.27	0.23	26	0.26	0.22	0.24
12	0.29	0.21	0.21	27	0.21	0.27	0.3
13	0.28	0.25	0.26	28	0.26	0.2	0.23
14	0.3	0.21	0.29	29	0.21	0.26	0.21
15	0.27	0.23	0.21	30	0.28	0.29	0.26

TABLE 5
RESULTS FOR THE CASE OF L_{ik} OF TABLE 4

First rational function	Second rational function	Third rational function
$\xi = 1.149994$		
$\delta_1 = 0.454503$	$\delta_2 = 0.365866$	$\delta_3 = 0.329625$
$c_1 = 0.395222$	$c_2 = 0.318146$	$c_3 = 0.286632$
$\frac{c_1}{\delta_1} = 0.869565$	$\frac{c_2}{\delta_2} = 0.869565$	$\frac{c_3}{\delta_3} = 0.869565$
$a_{11} = -8.695698$	$a_{21} = 21.103837$	$a_{31} = -6.643322$
$a_{12} = -8.695698$	$a_{22} = -5.006116$	$a_{32} = 19.968660$
$a_{13} = 5.011477$	$a_{23} = -8.695698$	$b_{31} = 2.401095$
$b_{11} = 5.526923$	$b_{21} = 8.876023$	$b_{32} = 13.764980$
$b_{12} = -5.526932$	$b_{22} = -8.876023$	$b_{33} = -8.695698$
$b_{13} = 21.151163$		

CASE (a). In this first case, we take for values that L_{ik} equals unity, i.e., $L_{ik} = 1$ for every i and for every k.

Table 3 gives the final approximation results, while Figures 4 and 5 show the fitting results and the approximation errors reached, respectively. Note that in this case, the maximum approximation error is equal to

$$\underset{k=1,2,\ldots,K}{\text{max imum}} \left| G(x_k) - \sum_{i=1}^{3} L_{ik} H_i(x_k) \right| = 0.0689. \tag{24}$$

CASE (b). In the second case, the values of L_{ik}, which are given in Table 4, were randomly generated in the interval [0.25, 0.35]. The solution of the approximation problem gives the results of Table 5. Moreover, Figures 6 and 7 depict the fitting results and the approximation errors, respectively. As is easily observed, in comparison with Case (a), Case (b) leads to greater approximation errors. Specifically, the maximum approximation error for Case (b) is equal to

$$\underset{k=1,2,\ldots,K}{\text{maximum}} \left| G(x_k) - \sum_{i=1}^{3} L_{ik} H_i(x_k) \right| = 2.3519. \tag{25}$$

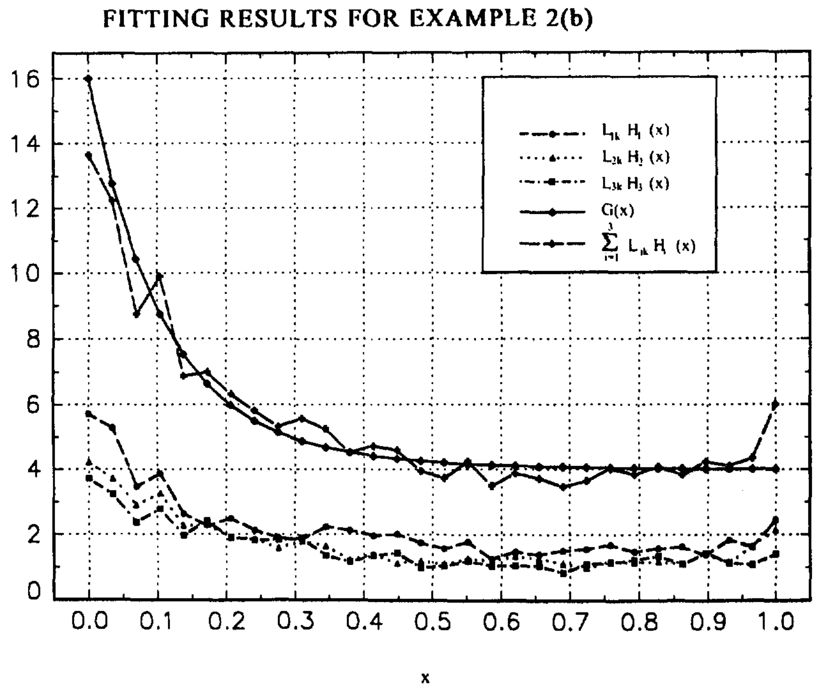

FIG. 6. Approximation of $G(x) = 4 + 10e^{-10x} + 2e^{-5x}$ by three rational functions and with L_{ik} of Table 4.

4. CONCLUSIONS

This paper has introduced a method for the optimum approximation of multi-real rational functions via linear programming. The approximation problem initially is not linear but it is transformed to a linear form by using appropriate transformations. The final formulation of the optimization problem includes suitable linear constraints that always ensure that the global minimum exists. Since the number of constraints is greater than the number of variables, the linear approximation problem is solved through its dual using the RSA.

It is important to notice that it is possible to use nonlinear formulation and nonlinear optimization techniques to solve a similar problem. But nonlinear techniques are associated with well-known disadvantages [10]. On the other hand, the linear programming formulation of the approximation problem, as it is described in the proposed paper, is the only method available in the literature that permits the simultaneous approximation of

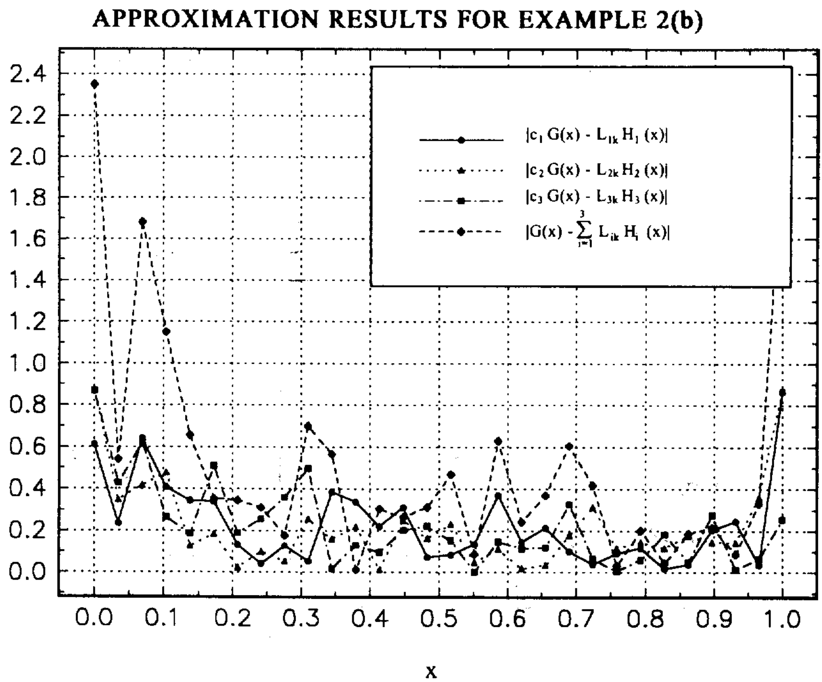

FIG. 7. Approximation errors for Example 2(b).

multi-real rational functions through linear programming. The usefulness of the proposed formulation is obvious, especially for engineering design problems. The two examples presented clearly illustrate the usefulness and effectiveness of the method.

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